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On Hermite Interpolation in Normed Vector Spaces

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1. INTRODUCTION

The object of one-dimensional Hermite interpolation is to find a polynomial P_M of degree at most M that satisfies the conditions:

$$P_M^{(k)}(x_i) = a_i^k \qquad (0 \leqslant i \leqslant m, 0 \leqslant k \leqslant \alpha_i), \tag{1.1}$$

where the $m + 1$ distinct points $x_i \in \mathbb{R}$, the $m + 1$ nonnegative integers $\alpha_i \geqslant 0$, satisfying $M + 1 = \sum_{i=0}^{m} (\alpha_i + 1)$, and the arbitrary real data a_i^k $(0 \leqslant i \leqslant m, 0 \leqslant k \leqslant \alpha_i)$ are given. Now let $x_0, x_1, ..., x_m$ be $m + 1$ distinct points of a normed vector space E. We shall prove that there always exists a continuous polynomial $P_M: E \to E$ of degree at most M which satisfies (1.1), where now $P_M^{(k)}(x_i)$ is the kth Fréchet derivative of P_M at x_i, and the $m + 1$ nonnegative integers $\alpha_i \geqslant 0$, satisfying $M + 1 = \sum_{i=0}^{m} (\alpha_i + 1)$, as well as the k-linear continuous symmetric mappings $a_i^k \in \mathscr{L}_s^k(E; E)$ $(0 \leqslant i \leqslant m, 0 \leqslant k \leqslant \alpha_i)$ are given. P. M. Prenter [6] has solved this problem for $\alpha_i = 0$ and $\alpha_i = 1$ $(0 \leqslant i \leqslant m)$, i.e., the Lagrange and the osculatory Hermite interpolation case. Our solution $P_M: E \to E$ will be composed of the mappings $l_i: E \to \mathscr{L}(E; E) = \mathscr{L}_s^1(E; E)$ which were constructed by Prenter [6].

The one-dimensional Hermite interpolation problem is always solvable by a uniquely determined interpolation polynomial $P_M: \mathbb{R} \to \mathbb{R}$ of degree $\leqslant M$ (see M. Müller [5]). Let the $m + 1$ functions L_i $(0 \leqslant i \leqslant m)$ be defined by

$$L_i: \mathbb{R} \ni x \mapsto \prod_{\substack{j=0 \\ j \neq i}}^{m} \left(\frac{x - x_j}{x_i - x_j} \right)^{\alpha_j + 1} \in \mathbb{R}. \tag{1.2}$$

Some transformations of the polynomial P_M, which is given by M. Müller [5], yield

THEOREM 1.1. *The interpolation polynomial $P_M: \mathbb{R} \to \mathbb{R}$ has the form*

$$P_M(x) = \sum_{i=0}^{m} \sum_{k=0}^{\alpha_i} \frac{1}{k!} (x - x_i)^k \cdot B_{ik}(x) \cdot L_i(x) \cdot a_i^k,$$

where

$$B_{ik}(x) = 1 + \sum_{\sigma=1}^{\alpha_i - k} (-1)^\sigma \sum_{m_0=0}^{\alpha_i-(k+\sigma)} \sum_{\substack{m_1,\ldots,m_\sigma \in \mathbb{N} \\ m_1+\cdots+m_\sigma = m_0+\sigma}}$$

$$\times \frac{1}{m_1! \cdot \cdots \cdot m_\sigma!} \prod_{\rho=1}^{\sigma} (L_i^{(m_\rho)}(x_i) \cdot (x - x_i)^{m_\rho}).$$

$L_i^{(m_\rho)}(x_i)$ *is the* m_ρth *derivative of* L_i *at the point* x_i.

2. Preliminary Notes

Let E and F be vector spaces (over the field $\mathbb{R}$ or $\mathbb{C}$) and $n \geqslant 1$ an integer. Then the mapping $\Delta_n: E \to E^n$ is defined by

$$\Delta_n: E \ni x \mapsto (x,\ldots, x) \in E^n.$$

If g maps E^n into F, then we shall write for abbreviation $g(x^n)$ instead of $(g \circ \Delta_n)(x)$. Moreover let $\Theta_n: E^n \to F$ be the zero linear operator from E^n to F.

A mapping $f: E \to F$ is called a *homogeneous polynomial* (on E into F) *of degree n*, if there exists an n-linear mapping $f_n: E^n \to F$ satisfying $f_n \neq \Theta_n$ and $f = f_n \circ \Delta_n$. Let $f: E \to F$ be a constant mapping and $\operatorname{Im} f = \{f_0\}$; in this case, we shall call f a *homogeneous polynomial of degree* 0, and we shall substitute $f(x^0)$ for f_0.

A mapping $f: E \to F$ is called a *polynomial* (on E into F) *of degree* $\leqslant M$, if there exist an integer $M \geqslant 0$ and a set $T \subset \{0,\ldots, M\}$ $(T \neq \varnothing)$ such that

$$f = \sum_{t \in T} f_t, \tag{2.1}$$

where each f_t is a homogeneous polynomial of degree t. If E and F are normed vector spaces and if f has a representation of the form (2.1), where the f_t are continuous homogeneous polynomials, then we shall call f a *continuous polynomial of degree* $\leqslant M$.

Remark 2.1 will show how to construct a new polynomial out of given polynomials $f_i: E \to E_i$ (see H. Cartan [1]).

Remark 2.1. Let $s + 2$ vector spaces $E, E_1,\ldots, E_s$ and F be given; moreover let $f_i: E \to E_i$ $(1 \leqslant i \leqslant s)$ be polynomials of degree $\leqslant p_i$. If $u: E_1 \times \cdots \times E_s \to F$ is an s-linear mapping, then the mapping $f: E \to F$ defined by

$$f(x) = u(f_1(x),\ldots, f_s(x))$$

is a polynomial (on E into F) of degree $\leqslant p = \sum_{i=1}^{s} p_i$. If the vector spaces

$E, E_1, \ldots, E_s$ and F are normed and if the s polynomials f_i and the s-linear mapping u are continuous, then f is a continuous polynomial of degree $\leqslant p$.

If $f\colon E \to F$ is a K-times Fréchet-differentiable mapping between the normed vector spaces E and F, then the kth Fréchet derivative of f at $z_0 \in E$ $(0 \leqslant k \leqslant K)$ can be considered as belonging to $\mathscr{L}_s^k(E; F)$, the vector space of all k-linear continuous symmetric mappings of E into F. We obtain

$$f^{(k)}(z_0)(y^k) = (f^{(k)}(z_0) \circ \Delta_k)(y)$$

and

$$f^{(0)}(z_0)(y^0) = f(z_0).$$

If $g_1, \ldots, g_s$ are elements of $\mathscr{L}(E; E)$, then we shall write

$$\prod_{i=1}^{s} g_i \quad \text{instead of} \quad g_1 \circ \cdots \circ g_s,$$

the composition of the s linear continuous mappings g_i.

In the following we will use a generalization of Leibniz's formula.

THEOREM 2.2. *Let the $s + 2$ normed vector spaces $E, E_1, \ldots, E_s$ and F be given, and the mappings $f_i\colon E \to E_i$ $(1 \leqslant i \leqslant s)$ be K-times differentiable at $z_0 \in E$ and $u\colon E_1 \times \cdots \times E_s \to F$ an s-linear continuous mapping. Then the* kth *Fréchet derivative $(k \in \{0, \ldots, K\})$ of the mapping*

$$f = u \circ (f_1, \ldots, f_s)\colon E \ni x \mapsto u(f_1(x), \ldots, f_s(x)) \in F$$

at the point $z_0 \in E$ is given by

$$f^{(k)}(z_0)\colon (y_1, \ldots, y_k) \mapsto \sum_{\substack{l_1, \ldots, l_s \geqslant 0 \\ l_1 + \cdots + l_s = k}} \frac{1}{l_1! \cdot \cdots \cdot l_s!}$$

$$\times \sum_{\Pi \in \gamma_k} u(f_1^{(l_1)}(z_0)(y_{\pi(1)}, \ldots, y_{\pi(l_1)}), \ldots, f_s^{(l_s)}(z_0)(y_{\pi(n_{s-1}+1)}, \ldots, y_{\pi(k)})).$$

Here γ_k is the group of all permutations of the set of indices $\{1, \ldots, k\}$ and $n_i = \sum_{j=1}^{i} l_j$ $(1 \leqslant i \leqslant s)$.

3. EXISTENCE AND CONSTRUCTION OF A HERMITE INTERPOLATION POLYNOMIAL P_M

Let E be a normed vector space. The points $x_i \in E$, the nonnegative integers $\alpha_i \geqslant 0$ and the arbitrary elements $a_i^k \in \mathscr{L}_s^k(E; E)$ are given as in the first section. We shall prove

THEOREM 3.1. *There exists a continuous polynomial P_M: $E \to E$ of degree $\leqslant M$ satisfying the Hermite interpolation conditions*

$$P_M^{(k)}(x_i) = a_i^k,$$

where $0 \leqslant i \leqslant m$ *and* $0 \leqslant k \leqslant \alpha_i$.

If we call the vector space of all continuous polynomials (on E into E) of degree $\leqslant M$ Π_M, then we have solved our Hermite interpolation problem, if we can find $M + 1$ polynomials $Q_{ik} \in \Pi_M$ $(0 \leqslant i \leqslant m, 0 \leqslant k \leqslant \alpha_i)$ such that the following conditions are satisfied:

$$Q_{ik}^{(l)}(x_i) = \begin{cases} a_i^k & \text{if } l = k \quad (3.1) \\ \Theta_l & \text{if } l \neq k \text{ and } 0 \leqslant l \leqslant \alpha_i \quad (3.2) \end{cases}$$

$$Q_{ik}^{(l)}(x_j) = \Theta_l \qquad \text{if } j \neq i, \ 0 \leqslant j \leqslant m, \text{ and } 0 \leqslant l \leqslant \alpha_j. \quad (3.3)$$

Then

$$P_M = \sum_{i=0}^{m} \sum_{k=0}^{\alpha_i} Q_{ik} \tag{3.4}$$

solves our Hermite interpolation problem. The polynomial P_M of Theorem 1.1 has such a representation in the case $E = \mathbb{R}$.

We shall construct mappings L_i: $E \to \mathscr{L}(E; E)$ corresponding to the functions L_i: $\mathbb{R} \to \mathbb{R}$ (see (1.2)). The starting point is the following lemma (see Prenter [6]).

LEMMA 3.2. *If $x_0, \ldots, x_m$ are $m + 1$ distinct points of a normed vector space E, then there exist continuous polynomials l_i: $E \to \mathscr{L}(E; E)$ of degree $\leqslant m$ $(0 \leqslant i \leqslant m)$ satisfying*

$$l_i(x_j) = \delta_{ij} \cdot \mathrm{id}_E \qquad (0 \leqslant i, j \leqslant m).$$

Here id_E is the identity map from E to E, defined by $\mathrm{id}_E(x) = x$ for every $x \in E$. δ_{ij} is the Kronecker symbol, and $0 \cdot \mathrm{id}_E$ means the element $\Theta_1 \in \mathscr{L}(E; E)$.

Now we construct the mappings L_i. Let the points $x_0, \ldots, x_m$ be given. Moreover let $\alpha_0, \ldots, \alpha_m$ be nonnegative integers satisfying $M = \sum_{i=0}^{m} (\alpha_i + 1) - 1$. We define the following sets of indices:

For fixed $i \in \{0, \ldots, m\}$ and $N = \{0, \ldots, m\}$

$$I := N \setminus \{i\}, \qquad k_i := \max\{\alpha_j ; j \in I\}$$

and

$$I_\nu := \{\mu \in I; \alpha_\mu \geqslant \nu\} \qquad (0 \leqslant \nu \leqslant k_i).$$

Then

$$I = I_0 \supset I_1 \supset \cdots \supset I_{k_i-1} \supset I_{k_i}.$$

According to Lemma 3.2 we construct the mappings $l_{i\nu}: E \to \mathscr{L}(E; E)$ $(0 \leqslant \nu \leqslant k_i)$ corresponding to x_μ, $\mu \in I_\nu \cup \{i\}$. Then

$$l_{i\nu}(x_{j_\nu}) = \delta_{ij_\nu} \cdot \mathrm{id}_E,$$

where $j_\nu \in I_\nu \cup \{i\}$. If we define L_i by

$$L_i(x) = \prod_{\nu=0}^{k_i} l_{i\nu}(x),$$

then

$$L_i(x_j) = \delta_{ij} \cdot \mathrm{id}_E,$$

where $0 \leqslant i,\ j \leqslant m$. Now, for $0 \leqslant i \leqslant m$ and $0 \leqslant k \leqslant \alpha_i$, let $B_{ik}: E \to \mathscr{L}(E; E)$ be defined by

$$B_{ik}(x) = \mathrm{id}_E + \sum_{\sigma=1}^{\alpha_i-k} (-1)^\sigma \sum_{m_0=0}^{\alpha_i-(k+\sigma)} \sum_{\substack{m_1,\dots,m_\sigma \in \mathbb{N} \\ m_1+\cdots+m_\sigma=m_0+\sigma}} \frac{1}{m_1! \cdot \cdots \cdot m_\sigma!}$$
$$\times \prod_{\rho=1}^{\sigma} L_i^{(m_\rho)}(x_i)((x - x_i)^{m_\rho}).$$

This means, in the notations of Section 2, that the m_ρ-linear continuous symmetric mapping $L_i^{(m_\rho)}(x_i) \in \mathscr{L}_s^{m_\rho}(E; \mathscr{L}(E; E))$ is applied to $\Delta_{m_\rho}(x - x_i)$.

Now we can define the polynomials $Q_{ik} \in \Pi_M$. Let

$$Q_{i0}(x) = (B_{i0}(x) \circ L_i(x))(a_i^0)$$
$$Q_{i1}(x) = (a_i^1 \circ B_{i1}(x) \circ L_i(x))(x - x_i)$$

and

$$Q_{ik}(x) = \frac{1}{k!} \cdot a_i^k((x - x_i)^{k-1}, (B_{ik}(x) \circ L_i(x))(x - x_i)),$$

where $2 \leqslant k \leqslant \alpha_i$.

Before dealing with the conditions (3.1)–(3.3) in the next section, we shall show that the Q_{ik} are continuous polynomials of degree $\leqslant M$.

LEMMA 3.3. *For each $i = 0,\dots, m$ and each $k = 0,\dots, \alpha_i$ the mappings Q_{ik} are elements of Π_M.*

Proof. According to the construction the mappings $l_{i\nu}: E \to \mathscr{L}(E; E)$ are continuous polynomials of degree $\leqslant |I_\nu|$, where $|I_\nu|$ is the cardinality of I_ν. Then Remark 2.1 shows that $L_i: E \to \mathscr{L}(E; E)$ is a continuous polynomial of degree $\leqslant \sum_{\nu=0}^{k_i} |I_\nu| = \sum_{j=0, j\neq i}^{m} (\alpha_j + 1)$. Furthermore $B_{ik}: E \to \mathscr{L}(E; E)$ is a continuous polynomial of degree $\leqslant \alpha_i - k$, since

$$m_1 + \cdots + m_\sigma = m_0 + \sigma \leqslant \alpha_i - (k + \sigma) + \sigma = \alpha_i - k.$$

Since $T_{x_i}: E \ni x \mapsto x - x_i \in E$ $(0 \leqslant i \leqslant m)$ is an element of Π_1 and $g: E \ni x \mapsto a_i^0 \in E$ is an element of Π_0, repeated use of Remark 2.1 yields that $Q_{ik}: E \to E$ is a continuous polynomial of degree

$$\leqslant \sum_{\substack{j=0\\ j\neq i}}^{m} (\alpha_j + 1) + (\alpha_i - k) + k = \sum_{\substack{j=0\\ j\neq i}}^{m} (\alpha_j + 1) + \alpha_i = M.$$

4. Proof of the Interpolation Properties of P_M

First of all we prove the interpolation property (3.1). To do that we define mappings $C_{ik}: E \to \mathscr{L}(E; E)$ by

$$C_{i0}(x) = B_{i0}(x) \circ L_i(x)$$

and

$$C_{ik}(x) = (B_{ik}(x) \circ L_i(x))(x - x_i)$$

for each $k = 1, \ldots, \alpha_i$. As a result of Theorem 2.2 we have the following

Remark 4.1. Let l be an integer $\geqslant 1$, $(y_1, \ldots, y_l) \in E^l$ and $z_0 \in E$. Then we have

$$Q_{i0}^{(l)}(z_0)(y_1, \ldots, y_l) = (C_{i0}^{(l)}(z_0)(y_1, \ldots, y_l))(a_i^0) \tag{4.1}$$

$$Q_{i1}^{(l)}(z_0)(y_1, \ldots, y_l) = a_i^1 \circ (C_{i1}^{(l)}(z_0)(y_1, \ldots, y_l)) \tag{4.2}$$

$$Q_{ik}^{(l)}(z_0)(y_1, \ldots, y_l) = \frac{1}{k!} \sum_{\substack{l_1, \ldots, l_k \geqslant 0\\ l_1 + \cdots + l_k = l}} \frac{l}{l_1! \cdot \cdots \cdot l_k!}$$

$$\times \sum_{\Pi \in \gamma_l} a_i^k (f_1^{(l_1)}(z_0)(\ldots), \ldots, f_{k-1}^{(l_{k-1})}(z_0)(\ldots), C_{ik}^{(l_k)}(z_0)(\ldots)) \tag{4.3}$$

where $f_\lambda: E \ni x \mapsto x - x_i \in E$ $(1 \leqslant \lambda \leqslant k - 1$ and $2 \leqslant k \leqslant \alpha_i)$.

LEMMA 4.2. *For each* $i = 0, \ldots, m$ *and each* $k = 0, \ldots, \alpha_i$ *we have*

$$Q_{ik}^{(k)}(x_i) = a_i^k.$$

Proof.

(a) For $k = 0$ we obtain, based on our definitions,

$$Q_{i0}(x_i) = (B_{i0}(x_i) \circ L_i(x_i))(a_i^0) = (\mathrm{id}_E \circ \mathrm{id}_E)(a_i^0) = a_i^0.$$

(b) For $k = 1$, by (4.2), we have

$$Q_{i1}^{(1)}(x_i) = a_i^1 \circ C_{i1}^{(1)}(x_i).$$

We define $g_1: E \to \mathscr{L}(E; E)$ by $g_1(x) = B_{i1}(x) \circ L_i(x)$, $g_2: E \to E$ by $g_2(x) = x - x_i$ and $u: \mathscr{L}(E; E) \times E \to E$ by $u(h, x) = h(x)$. Using Theorem 2.2 we get in consequence of $g_2^{(1)}(z_0) = \mathrm{id}_E$ $(z_0 \in E)$

$$C_{i1}^{(1)}(x_i) = B_{i1}(x_i) \circ L_i(x_i) \circ \mathrm{id}_E\,;$$

therefore

$$Q_{i1}^{(1)}(x_i) = a_i^1.$$

(c) For $k = 2, \ldots, \alpha_i$ we look at the representation (4.3) where $z_0 = x_i$. Let $(l_1, \ldots, l_k)$ be a fixed k-tuple satisfying $l_1 + \cdots + l_k = k$.

(i) Suppose $l_k \geqslant 2$. Then there exists a λ $(1 \leqslant \lambda \leqslant k - 1)$ where $l_\lambda = 0$. Therefore the corresponding term vanishes because of $f_\lambda(x_i) = 0$.

(ii) Suppose $l_\lambda \geqslant 2$ for some $\lambda \in \{1, \ldots, k - 1\}$. Then $f_\lambda^{(\nu)}(z_0) = \Theta_\nu$ for arbitrary $z_0 \in E$ and $\nu \geqslant 2$. Thus the corresponding term vanishes.

(iii) We still have to look at the term where $(l_1, \ldots, l_k) = (1, \ldots, 1)$. For $\lambda = 1, \ldots, k - 1$ and arbitrary $z_0 \in E$ it is valid that $f_\lambda^{(1)}(z_0) = \mathrm{id}_E$. Moreover $C_{ik}^{(1)}(x_i) = \mathrm{id}_E$. Therefore we obtain

$$a_i^k(y_{\pi(1)}, \ldots, y_{\pi(k)}).$$

The statement for $k = 2, \ldots, \alpha_i$ follows from (i) to (iii), since the mappings a_i^k are symmetric.

The property (3.3) follows from the following

LEMMA 4.3. *Let* $i, j \in \{0, \ldots, m\}$, $i \neq j$, *be given. Then*

$$\left.\begin{aligned} L_i^{(l)}(x_j) &= \Theta_{l+1} \\ C_{i0}^{(l)}(x_j) &= \Theta_{l+1} \end{aligned}\right\} \qquad (0 \leqslant l \leqslant \alpha_j)$$

and

$$C_{ik}^{(l)}(x_j) = \Theta_l$$

for $l = 0, \ldots, \alpha_j$ *and* $k = 1, \ldots, \alpha_i$.

The proof of Lemma 4.3 follows immediately from Theorem 2.2. The property (3.2) for $0 \leqslant l < k$ is formulated in the following lemma, and can also be proved without difficulties.

LEMMA 4.4. *For $i = 0,..., m$, $k = 0,..., \alpha_i$ and $0 \leqslant l < k$ there is*

$$Q_{ik}^{(l)}(x_i) = \Theta_l .$$

Proof. For $k = 1$ the statement holds. Now suppose $k \geqslant 2$. We look at (4.3). Since $l < k$ there exists for each k-tuple $(l_1 ,..., l_k)$ at least one $\lambda \in \{1,..., k\}$ for which $l_\lambda = 0$. If $\lambda = k$, then the corresponding term vanishes because of $C_{ik}(x_i) = 0$; if $\lambda \in \{1,..., k - 1\}$, then it vanishes since $f_\lambda(x_i) = 0$.

The property (3.2) for the remaining indices $k < l \leqslant \alpha_i$ follows from Lemma 4.5. Here we omit the long, but technical proof of that lemma.

LEMMA 4.5. *For $i = 0,..., m$, $\alpha_i \geqslant 1$ and $l = 1,..., \alpha_i$ we have*

$$C_{i0}^{(l)}(x_i) = \Theta_{l+1} ;$$

for $i = 0,..., m$, $\alpha_i \geqslant 2$ and $1 \leqslant k < l \leqslant \alpha_i - k + 1$ we obtain

$$C_{ik}^{(l)}(x_i) = \Theta_l .$$

Theorem 3.1 yields

COROLLARY 4.6. *Let $f: E \supset V \to E$ be α-times Fréchet-differentiable in the open set $V \subset E$ and let $z_0 \in V$. Choosing $m = 0$ and $a_0{}^k := f^{(k)}(z_0)$ for each $k = 0,..., \alpha$, then the Hermite interpolation polynomial of degree $\leqslant \alpha$ (at the point z_0) is the uniquely determined Taylor polynomial of degree $\leqslant \alpha$ at the point z_0.*

5. QUESTIONS CONCERNING UNIQUENESS

If E is the product of the n normed vector spaces E_r, i.e., $E = \prod_{r=1}^n E_r$, $pr_r: E \to E_r$ is the canonical projection and $j_r: E_r \to E$ $(1 \leqslant r \leqslant n)$ is the canonical injection, then we have for a k-linear continuous mapping $u \in \mathscr{L}^k(E; E)$

$$u = \sum_{r=1}^{n} j_r \circ pr_r \circ u =: \sum_{r=1}^{n} j_r \circ u_r ,$$

where $u_r \in \mathscr{L}^k(E; E_r)$ is a k-linear continuous mapping $(1 \leqslant r \leqslant n)$. Let

$T: E \to E$ be a continuous polynomial of degree $\leqslant M$. Then there exist n continuous polynomials $T_r: E \to E_r$ of degree $\leqslant M$ with

$$T = \sum_{r=1}^{n} j_r \circ T_r .$$

If $P_M = \sum_{i=0}^{m} \sum_{k=0}^{\alpha_i} Q_{ik}$ fulfills the interpolation conditions (3.1)–(3.3), then for the corresponding indices i, j, k, l it follows:

$$Q_{ik}^{(l)}(x_j) = \sum_{r=1}^{n} j_r \circ (pr_r \circ Q_{ik})^{(l)}(x_j)$$

$$= \begin{cases} \sum_{r=1}^{n} j_r \circ a_{ir}^{k} & \text{for } j = i \text{ and } l = k \\ \sum_{r=1}^{n} j_r \circ \Theta_{lr} & \text{for } j \neq i \text{ or } l \neq k. \end{cases}$$

Thus there exist n continuous polynomials $P_r := pr_r \circ P: E \to E_r$ satisfying the interpolation conditions (3.1)–(3.3) for arbitrarily given $a_{ir}^{k} \in \mathscr{L}_s^{k}(E; E_r)$ and the zero linear operators $\Theta_{lr} \in \mathscr{L}^{l}(E; E_r)$ $(r = 1,..., n)$.

Now we look at the normed vector space $E = \mathbb{R}^n$. Then there always exists for $m + 1$ arbitrarily given points $x_i \in \mathbb{R}^n$ and for $m+1$ nonnegative integers α_i, satisfying $M + 1 = \sum_{i=0}^{m} (\alpha_i + 1)$, a continuous polynomial $P_M: \mathbb{R}^n \to \mathbb{R}$ of degree $\leqslant M$ with

$$P_M^{(k)}(x_i) = a_i^{k} \qquad (0 \leqslant i \leqslant m, 0 \leqslant k \leqslant \alpha_i),$$

where the k-linear symmetric mappings $a_i^k \in \mathscr{L}_s^{k}(\mathbb{R}^n; \mathbb{R})$ are given. For $m = 0$ and arbitrarily given mappings $a_0^k \in \mathscr{L}_s^{k}(\mathbb{R}^n; \mathbb{R})$ $(0 \leqslant k \leqslant \alpha_0)$ we get a uniquely determined Hermite interpolation polynomial.

Thus we have a generalization of the one-dimensional Hermite interpolation, which doesn't possess for $n \geqslant 2$ and $m \geqslant 1$ a uniquely determined solution, in general. We shall illustrate these facts on the basis of the number of the interpolation conditions and of the number of the coefficients of a polynomial $P_M: \mathbb{R}^n \to \mathbb{R}$ of degree $\leqslant M$. A homogeneous polynomial $p: \mathbb{R}^n \to \mathbb{R}$ of degree k possesses $\binom{k+n-1}{k}$ coefficients (see C. Coatmelec [2]). Therefore a polynomial $P_M: \mathbb{R}^n \to \mathbb{R}$ of degree M consists of

$$\sum_{k=0}^{M} \binom{k+n-1}{k} = \binom{M+n}{M}$$

coefficients. Thus the Taylor polynomial of degree α_0 is described by $\binom{\alpha_0+n}{\alpha_0}$ real numbers, which determine the k-linear symmetric mappings $a_0^k \in \mathscr{L}_s^k(\mathbb{R}^n; \mathbb{R})$ $(0 \leqslant k \leqslant \alpha_0)$. For $m = 1$ and the nonnegative integers α_0, α_1 we obtain by the k-linear symmetric mappings a_i^k $(i = 0, 1, 0 \leqslant k \leqslant \alpha_i)$ altogether

$$\binom{\alpha_0 + n}{\alpha_0} + \binom{\alpha_1 + n}{\alpha_1}$$

conditions for the determination of

$$\binom{\alpha_0 + \alpha_1 + n + 1}{\alpha_0 + \alpha_1 + 1}$$

unknowns. For $n \geqslant 2$ we have

$$\begin{aligned}
\binom{\alpha_0 + \alpha_1 + n + 1}{\alpha_0 + \alpha_1 + 1} &= \sum_{k=0}^{\alpha_0+\alpha_1+1} \binom{k + n - 1}{k} \\
&= \sum_{k=0}^{\alpha_0} \binom{k + n - 1}{k} + \sum_{k=\alpha_0+1}^{\alpha_0+\alpha_1+1} \binom{k + n - 1}{k} \\
&> \binom{\alpha_0 + n}{\alpha_0} + \sum_{l=0}^{\alpha_1} \binom{l + n - 1}{l} \\
&= \binom{\alpha_0 + n}{\alpha_0} + \binom{\alpha_1 + n}{\alpha_1}
\end{aligned}$$

because of

$$\binom{\alpha_0 + n + l}{\alpha_0 + 1 + l} > \binom{n - 1 + l}{l} \qquad \text{for} \quad l = 0, \ldots, \alpha_1.$$

Thus we get for $n \geqslant 2$ less interpolation conditions than unknowns and therefore no unique solution of the Hermite interpolation problems for $m \geqslant 1$.

If we consider certain tensor product Hermite interpolation problems, i.e., choosing certain distributions of points in $\mathbb{R}^n$, prescribing partial derivatives at the points $x_i \in \mathbb{R}^n$ and interpolating with polynomials $p \in \bigotimes_{r=1}^{n} \Pi_{M_r}$, the tensor product of n spaces of polynomials in one variable, then we obtain uniquely determined Hermite interpolation polynomials (see W. Haussmann [3, 4]).

References

1. H. Cartan, "Calcul Différentiel," Hermann, Paris, 1967.
2. C. Coatmelec, Approximation et interpolation des fonctions différentiables de plusieurs variables, *Ann. Sci. Ecole Norm. Sup.* (*3*) **83** (1966), 271–341.
3. W. Haussmann, Tensorprodukte und mehrdimensionale Interpolation, *Math. Z.* **113** (1970), 17–23.
4. W. Haussmann, Tensorproduktmethoden bei mehrdimensionaler Interpolation, *Math. Z.* **124** (1972), 191–198.
5. M. Müller, Über Interpolation mittels ganzer rationaler Funktionen, *Math. Z.* **62** (1955), 292–309.
6. P. M. Prenter, Lagrange and Hermite interpolation in Banach spaces, *J. Approximation Theory* **4** (1971), 419–432.